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On canonical formalism in field theory with derivatives of higher order—canonical transformations

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Abstract. For a canonical formalism with derivatives of higher order, the corresponding theory of canonical transformations is given in the most general case for classical and covariant field theory. Relations with the generating functional, infinitesimal transformations, Hamilton–Jacobi method, Lagrange and Poisson brackets, as well as integral invariants of the first and higher orders and the corresponding Liouville theorem are considered.

1. Introduction

Over thirty years ago, Bopp and Podolski attempted a generalisation of electrodynamics, based on the Lagrangian with second-order derivatives. Then an interest was taken in the generalisation of the Hamiltonian canonical formalism for the case when the derivatives of arbitrary order appear in the Lagrangian. The corresponding canonical equations have been obtained by Ostrogradski (1850) and de Donder (1935), using the calculus of variations, and independently by Koestler and Smith (1965) in analytical mechanics. Also Rodrigues and Rodrigues (1970) have formulated the corresponding canonical transformations in this formalism, correcting the definition of the Poisson bracket given by the mentioned authors so that this is always invariant, and obtained the corresponding Hamilton–Jacobi equation on the basis of canonical transformations.

In classical field theory Chang (1948) was the first to formulate the canonical formalism, but in an implicit manner; de Wet (1948) introduced the generalised momenta explicitly, starting from the variation of the action, and obtaining the corresponding canonical equations. Independently of these latter results Borneas (1960) found the corresponding Hamiltonian by transforming the Lagrangian equations into a convenient form, and Thielheim (1967) derived expressions for the density of energy, momentum and angular momentum in field theory, as well as the canonical equations for such fields.

Nevertheless, their definitions of the generalised momenta showed certain deficiencies, and only Coelho de Souza and Rodrigues (1969) defined these quantities in a way completely analogous to that in analytical mechanics, and thus obtained the corresponding generalised Hamiltonian equations. They were the first and only authors to have investigated canonical transformations in classical field theory, and to have formulated the corresponding generalised Poisson brackets, proving their invariance with respect to the canonical transformations.

Canonical formalism in field theory may be expressed either directly in a covariant manner, by using four space–time coordinates, or in a parametric manner using the space-like surfaces. In the general case, when the Lagrangian depends on derivatives of arbitrary order, de Donder (1935) used the calculus of variations to derive the corresponding equations of the extremals for several independent variables in the canonical form, which can be interpreted as the covariant canonical equations in the direct formalism.

The corresponding canonical transformations were studied only in the usual case by Weiss (1938), who based them on the space-like surfaces, and in the direct formalism by Good (1954) and Liotta (1956). Nevertheless, their most important results cannot be considered correct, since the Poisson bracket introduced by Good in the general case is not invariant and does not exhibit any characteristics of these brackets. Freistadt (1959) defined the Poisson bracket with the aid of the space-like surfaces and on this basis the canonical transformations were studied in the form of the functionals by the author (Mouchitzky† 1965, Mušicki 1968), who obtained the corresponding differential and integral invariants.

In this general case de Wet (1948), Thielheim (1967) and Borneas (1969) studied the general variation of the action and conservation laws in direct formalism, but without any connection with canonical transformations, which were studied only in classical field theory. As far as the author knows, there are no published papers devoted to the problem of the covariant formulation of the canonical transformations in field theory.

2. Classical field theory

2.1. Generalised Hamiltonian equations in field theory

Let us consider a physical field defined by r field functions $\psi_k(x_i, t)$ and assume that it can be described by a Lagrangian in the form of a functional:

$$L = L[\psi_k, \dot{\psi}_k, \ldots, \psi_k^{(s)}; t] \equiv \int \mathscr{L} \,\mathrm{d}V. \tag{2.1}$$

The corresponding Lagrangian equations, equivalent to the Hamiltonian principle, may be written as

$$\frac{\delta L}{\delta \psi_k} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta L}{\delta \dot{\psi}_k} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\delta L}{\delta \ddot{\psi}_k} - \ldots + (-1)^s \frac{\mathrm{d}^s}{\mathrm{d}t^s}\frac{\delta L}{\delta \psi_k^{(s)}} = 0,$$

where the calculus of functionals (Volterra 1959) is used, or more concisely

$$\delta W/\delta \psi_k = 0, \qquad W \equiv \int_{t_0}^{t_1} L \,\mathrm{d}t. \tag{2.2}$$

Introducing the generalised momenta by

$$\pi_{k/m} = \frac{\delta W}{\delta \psi_k^{(m)}} = \sum_{j=0}^{s-m} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}t^j}\frac{\delta L}{\delta \psi_k^{(j+m)}}, \tag{2.3}$$

† This is a transliteration of Mušicki.

Coelho de Souza and Rodrigues (1969) obtained the corresponding canonical equations in the form

$$\dot{\pi}_{k/m} = -\frac{\delta H}{\delta \psi_k^{(m-1)}}, \qquad \dot{\psi}_k^{(m-1)} = \frac{\delta H}{\delta \pi_{k/m}}, \qquad (k = 1, 2, \ldots, r;\ m = 1, 2, \ldots, s) \tag{2.4}$$

where the Hamiltonian is

$$H[\psi_k^{(m-1)}, \pi_{k/m};\ t] = \int \left(\sum_{k=1}^{r} \sum_{m=1}^{s} \pi_{k/m} \psi_k^{(m)} - \mathscr{L} \right) \mathrm{d}V. \tag{2.5}$$

These are the generalised Hamiltonian equations for fields and they represent a system of $2rs$ first-order differential equations with canonical variables $\psi_k^{(m-1)}$ and $\pi_{k/m}$ as unknown functions. The Hamiltonian may be formed in a manner analogous to that in particle mechanics, by eliminating only the higher-order derivatives.

In certain cases these systems can be reduced to standard systems, namely when

$$L[\psi_k, \dot{\psi}_k, \ldots, \psi_k^{(s)};\ t] = L_0[\psi_k, \dot{\psi}_k;\ t] + \mathrm{d}F[\psi_k, \dot{\psi}_k, \ldots, \psi_k^{(s-1)};\ t]/\mathrm{d}t. \tag{2.6}$$

Then in each point

$$\frac{\delta L}{\delta \psi_k^{(s)}} = \frac{\delta F}{\delta \psi_k^{(s-1)}}, \qquad \frac{\delta^2 L}{\delta \psi_k^{(s)} \delta \psi_l^{(s)}} = 0,$$

from which follows

$$\Delta \equiv \left| \frac{\delta^2 L}{\delta \psi_k^{(s)} \delta \psi_l^{(s)}} \right| = 0, \tag{2.7}$$

so that all such reducible systems are degenerate in the sense of Dirac (1950).

2.2. Formulation of the canonical transformations

The canonical transformations were defined by the quoted authors as such transformations of the canonical variables

$$\bar{\psi}_k^{(m-1)} = F_{km}[\psi_l^{(n-1)}, \pi_{l/n};\ t], \qquad \bar{\pi}_{k/m} = G_{km}[\psi_l^{(n-1)}, \pi_{l/n};\ t] \tag{2.8}$$

which leave the form of the generalised Hamiltonian equations invariant. The theory of thus defined canonical transformations may be developed in analogy with analytical mechanics in the following manner.

With the Hamiltonian equations equivalent to the Hamiltonian principle, the necessary and sufficient condition for the transformation (2.8) to be canonical is

$$\int \left(\sum_k \sum_m \pi_{k/m}\, \mathrm{d}\psi_k^{(m-1)} - \mathscr{H}\, \mathrm{d}t \right) \mathrm{d}V = \int c(x) \left(\sum_k \sum_m \bar{\pi}_{k/m}\, \mathrm{d}\bar{\psi}_k^{(m-1)} - \bar{\mathscr{H}}\, \mathrm{d}t \right) \mathrm{d}V + \mathrm{d}G, \tag{2.9}$$

where $c(x)$ is any function of the space coordinates and G is the corresponding generating functional. Taking this as a functional of the type $G_1[\psi_k^{(m-1)}, \bar{\psi}_k^{(m-1)};\ t]$, one obtains

$$\pi_{k/m} = \frac{\delta G_1}{\delta \psi_k^{(m-1)}}, \qquad c\bar{\pi}_{k/m} = -\frac{\delta G_1}{\delta \bar{\psi}_k^{(m-1)}}, \qquad \bar{c}\bar{H} = H + \frac{\partial G_1}{\partial t}, \tag{2.10}$$

where $\bar{c}$ is the mean value of the function $c(x)$ in the domain V; while for the generating function of the type $G_2[\psi_k^{(m-1)}, \bar{\pi}_{k/m}; t]$:

$$\pi_{k/m} = \frac{\delta G_2}{\delta \psi_k^{(m-1)}}, \qquad c\bar{\psi}_k^{(m-1)} = \frac{\delta G_2}{\delta \bar{\pi}_{k/m}}, \qquad c\bar{H} = H + \frac{\partial G_2}{\partial t}. \tag{2.11}$$

If the generating functional G_1, or G_2, has the form of an integral whose integrand contains derivatives with respect to space coordinates, these relations represent a system of differential equations, which may be reduced to algebraic equations if the integrand does not contain these derivatives. In the general case they will form a system of functional differential equations, for example integro-differential equations. Solving the first group of these equations with respect to $\bar{\psi}_k^{(m-1)}$, or $\bar{\pi}_{k/m}$, and inserting the obtained solutions into the second group, one finds new canonical variables as functionals of the old ones.

Consider canonical transformations of the form

$$\bar{\psi}_k^{(m-1)} = \psi_k^{(m-1)} + \delta \psi_k^{(m-1)}, \qquad \bar{\pi}_{k/m} = \pi_{k/m} + \delta \pi_{k/m}. \tag{2.12}$$

Since

$$G_2^0[\psi_k^{(m-1)}, \bar{\pi}_{k/m}] = \int \sum_k \sum_m \psi_k^{(m-1)} \bar{\pi}_{k/m} \, \mathrm{d}V$$

yields an identical transformation, the generating functional may be taken in the form

$$G_2[\psi_k^{(m-1)}, \bar{\pi}_{k/m}; t] = \int \sum_k \sum_m \psi_k^{(m-1)} \bar{\pi}_{k/m} \, \mathrm{d}V + \epsilon G'[\psi_k^{(m-1)}, \bar{\pi}_{k/m}; t], \tag{2.13}$$

where ϵ is a small parameter. According to (2.11) one obtains approximately

$$\delta \psi_k^{(m-1)} = \epsilon \, \delta G' / \delta \pi_{k/m}, \qquad \delta \pi_{k/m} = -\epsilon \, \delta G' / \delta \psi_k^{(m-1)}. \tag{2.14}$$

For $G' = H$ and $\epsilon = \mathrm{d}t$ these relations give, with the aid of the generalised Hamiltonian equations,

$$\delta \psi_k^{(m-1)} = \mathrm{d}\psi_k^{(m-1)}, \qquad \delta \pi_{k/m} = \mathrm{d}\pi_{k/m}, \tag{2.15}$$

i.e. time evolution of the state of a field in any point may be represented as a series of the subsequent infinitesimal canonical transformations.

2.3. Hamilton–Jacobi method

If we consider a canonical transformation for which the new Hamiltonian is zero, the new canonical variables are independent of time, but are arbitrary functions of the space coordinates

$$\bar{\pi}_{k/m} = \alpha_{km}(x), \qquad \bar{\psi}_k^{(m-1)} = \beta_{km}(x). \tag{2.16}$$

In this case, by substituting all the generalised momenta in the Hamiltonian by $\delta S / \delta \psi_k^{(m-1)}$, the last relation (2.11) gives

$$\frac{\partial S}{\partial t} + H\left[\psi_k^{(m-1)}, \frac{\delta S}{\delta \psi_k^{(m-1)}}; t\right] = 0. \tag{2.17}$$

This is the corresponding Hamilton–Jacobi equation, which has the form of a functional differential equation. If we find a complete integral, the canonical variables may be obtained by solving the system of equations

$$\frac{\delta S[\psi_k^{(m-1)}, \alpha_{km}; t]}{\delta \psi_k^{(m-1)}} = \pi_{k/m}, \qquad \frac{\delta S[\psi_k^{(m-1)}, \alpha_{km}; t]}{\delta \alpha_{km}} = c\beta_{km}. \tag{2.18}$$

In the case when the functional S has the form of an integral, this will be a system of differential or algebraic equations, and in the general case a system of functional differential equations. Solving the second group of these equations with respect to $\psi_k^{(m-1)}$ and inserting the obtained solutions into the first group, one will obtain all the canonical variables as functionals of α_{ln}, β_{ln} and t.

2.4. *Lagrange and Poisson brackets*

The condition (2.9) can be transformed only into old variables if we substitute $\mathrm{d}\bar{\psi}_k^{(m-1)}$ by the corresponding functional differential according to (2.8)

$$\int \Big(\sum_k \sum_m P_{km}(x)\, \mathrm{d}\psi_k^{(m-1)} + \sum_k \sum_m Q_{km}(x)\, \mathrm{d}\pi_{k/m} + R(x)\, \mathrm{d}t\Big)\, \mathrm{d}V = \mathrm{d}G,$$

where P_{km}, Q_{km} and R are certain functions of both old and new canonical variables. By applying the conditions which make this expression a total differential, they may be written in a more concise form:

$$\begin{aligned} &\{\psi_k^{(m-1)}(x), \psi_l^{(n-1)}(x')\}_{\bar{\psi},\bar{\pi}} = 0, \qquad \{\pi_{k/m}(x), \pi_{l/n}(x')\}_{\bar{\psi},\bar{\pi}} = 0, \\ &\{\psi_k^{(m-1)}(x), \pi_{l/n}(x')\}_{\bar{\psi},\bar{\pi}} = (\delta_{kl}\delta_{mn}\delta(x-x'))/\tilde{c}, \end{aligned} \tag{2.19}$$

where the generalised Lagrange bracket is introduced:

$$\{u(x), v(x')\} = \int \sum_k \sum_m \left(\frac{\delta\psi_k^{(m-1)}(x'')}{\delta u(x)} \frac{\delta\pi_{k/m}(x'')}{\delta v(x')} - \frac{\delta\psi_k^{(m-1)}(x'')}{\delta v(x')} \frac{\delta\pi_{k/m}(x'')}{\delta u(x)}\right) \mathrm{d}V''. \tag{2.20}$$

These brackets are connected with the Poisson brackets by a relation analogous to that in analytical mechanics. With the aid of this relation and (2.19) one may also express these conditions in the form of the generalised Poisson brackets:

$$\begin{aligned} &[\psi_k^{(m-1)}(x), \psi_l^{(n-1)}(x')]_{\bar{\psi},\bar{\pi}} = 0, \qquad [\pi_{k/m}(x), \pi_{l/n}(x')]_{\bar{\psi},\bar{\pi}} = 0, \\ &[\psi_k^{(m-1)}(x), \pi_{l/n}(x')]_{\bar{\psi},\bar{\pi}} = \tilde{c}\delta_{kl}\delta_{mn}\delta(x-x'). \end{aligned} \tag{2.21}$$

If we start from the generalised Lagrange bracket in the new canonical variables and transform the corresponding functional derivatives to the old ones, this bracket will be reduced to the fundamentals, so that using (2.19) one finds

$$\{u(x), v(x')\}_{\bar{\psi},\bar{\pi}} = \{u(x), v(x')\}_{\psi,\pi}/\tilde{c}. \tag{2.22}$$

For the generalised Poisson bracket one obtains analogously

$$[u(x),\ v(x')]_{\bar{\psi},\bar{\pi}} = \tilde{c}[u(x), v(x')]_{\psi,\pi}, \tag{2.23}$$

which for $c = 1$ is identical with the result obtained by Coelho de Souza and Rodrigues (1969). So one may conclude that the generalised Lagrange and Poisson brackets remain invariant with respect to the canonical transformations up to the multiplier $1/\tilde{c}$, or $\tilde{c}$.

2.5. de Donder's relation and integral invariants

In order to present the states of a physical field geometrically, let us introduce the phase space whose elements are $\psi_k^{(m-1)}(x_i, t)$ and $\pi_{k/m}(x_i, t)$, considering the space coordinates x_i to be fixed parameters. If we consider one line in this phase space defined by

$$\psi_k^{(m-1)} = \psi_k^{(m-1)}(t, \alpha), \qquad \pi_{k/m} = \pi_{k/m}(t, \alpha), \tag{2.24}$$

and denote by δ' the variation arising from the change of the parameter α for $\delta t = 0$, we can introduce the quantity

$$j = \int \sum_k \sum_m \frac{\delta W}{\delta \psi_k^{(m)}} \delta' \psi_k^{(m-1)} \, \mathrm{d}V. \tag{2.25}$$

Starting from the identity, which comes from the definition of the functional derivative

$$\frac{\delta W}{\delta \psi_k^{(m-1)}} = \frac{\delta L}{\delta \psi_k^{(m-1)}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\delta W}{\delta \psi_k^{(m)}},$$

upon multiplication by $\delta' \psi_k^{(m-1)}$ and summation, we obtain

$$\frac{\mathrm{d}j}{\mathrm{d}t} = \delta' L - \int \sum_k \frac{\delta W}{\delta \psi_k} \delta' \psi_k \, \mathrm{d}V. \tag{2.26}$$

This relation represents the generalisation of one of the relations by de Donder (1935, p 98) and it corresponds to the central Lagrangian equation in analytical mechanics. If the Lagrangian equations (2.2) are satisfied, the last term vanishes, so that integrating this relation along the closed curve (2.24) gives the quantity

$$\mathcal{T}_1 \equiv \oint_L \int \sum_k \sum_m \frac{\delta W}{\delta \psi_k^{(m)}} \delta' \psi_k^{(m-1)} \, \mathrm{d}V = \oint_L \int \sum_k \sum_m \pi_{k/m} \delta' \psi_k^{(m-1)} \, \mathrm{d}V \tag{2.27}$$

which remains constant with time.

In the general case, when time also varies, we should rather start from the general variation of the action

$$\delta W = \left| \int \left(\sum_k \sum_m \pi_{k/m} \delta \psi_k^{(m-1)} - \mathcal{H} \delta t \right) \mathrm{d}V \right|_0^1.$$

If we imagine trajectories of the state drawn through every point of any closed curve L_0 in the extended phase space, then by integrating the above relation along this curve we obtain

$$\mathcal{T} \equiv \oint_{L_0} \int \left(\sum_k \sum_m \pi_{k/m} \delta \psi_k^{(m-1)} - \mathcal{H} \delta t \right) \mathrm{d}V = \oint_{L_0} \int \left(\sum_k \sum_m \pi_{k/m} \delta \psi_k^{(m-1)} - \mathcal{H} \delta t \right) \mathrm{d}V. \tag{2.28}$$

This is the corresponding Poincaré–Cartan integral and has the same value along all the curves on the surface formed by these trajectories.

In order to investigate the behaviour of this integral to the canonical transformations, let us start from the necessary and sufficient condition (2.9), where d will be substituted by δ, and integrate it along any closed curve. In this way, since $\oint \delta G = 0$, we find

$$\bar{\mathcal{T}} = \mathcal{T} / \tilde{c}, \tag{2.29}$$

i.e. the generalised Poincaré–Cartan integral is invariant to the canonical transformations up to the multiplier $1/\tilde{c}$.

2.6. *Absolute integral invariants of higher order*

The integral (2.27) may be transformed into a surface integral by the use of Stokes' theorem generalised to the functionals

$$\mathcal{T}_1 = \oint_L \int \sum_k \sum_m \pi_{k/m} \delta\psi_k^{(m-1)} \, dV = \int_S \int \sum_k \sum_m \delta\psi_k^{(m-1)} \delta\pi_{k/m} \, dV. \tag{2.30}$$

To obtain other absolute invariant integrals of higher order corresponding to Poincaré invariant integrals in analytical mechanics, it is necessary to avoid the non-denumerability of the set of functions $\psi_k(x_i, t)$, because of the continuous index x. Let us divide the domain V into a very large number N of cells, taking all values of the functions $\psi_k(x_i, t)$ to be equal in all points of the same cell. In this way the considered physical field is substituted by an equivalent system with Nr degrees of freedom.

Let us form now, in analogy with analytical mechanics for this generalised case, the following set of integrals:

$$\begin{aligned}
\mathcal{T}_1 &= \int_S \sum_x \sum_k \sum_m \delta\psi_k^{(m-1)}(x) \delta\pi_{k/m}(x) \\
\mathcal{T}_2 &= \int_S \int_S \sum_{x,x'} \sum_{k,k'} \sum_{m,m'} \delta\psi_k^{(m-1)}(x) \delta\psi_{k'}^{(m'-1)}(x') \delta\pi_{k/m}(x) \delta\pi_{k'/m'}(x') \\
&\vdots \\
\mathcal{T}_{rsN} &= \int_S \int_S \cdots \int_S \delta\psi_1(x_1) \ldots \delta\psi_1(x_N) \ldots \delta\psi_r^{(s-1)}(x_1) \ldots \delta\psi_r^{(s-1)}(x_N) \\
&\quad \times \delta\pi_{1/1}(x_1) \ldots \delta\pi_{1/1}(x_N) \ldots \delta\pi_{r/s}(x_1) \ldots \delta\pi_{r/s}(x_N).
\end{aligned} \tag{2.31}$$

Here Σ_x denotes the summation with respect to the mentioned cells and this summation is extended only over the terms with different indices, discrete and continuous. In the limit $N \to \infty$ the last integral may be interpreted as the volume of this phase space

$$\Delta\Gamma = \lim_{N\to\infty} \mathcal{T}_{rsN} = \lim_{N\to\infty} \int \prod \delta\psi_k^{(m-1)} \delta\pi_{k/m}. \tag{2.32}$$

By introducing the Gaussian parameters and developing the Jacobian of this transformation into second-order minors by generalisation of the corresponding relation in analytical mechanics (Mercier 1955), the element of any of these integrals may be written in the form:

$$\begin{aligned}
d\mathcal{T}_f &= \sum_{x'} \cdots \sum_{x^{(f)}} \sum_{k_1} \cdots \sum_{k_f} \sum_{m_1} \cdots \sum_{m_f} \frac{\partial(\psi_{k_1}^{(m_1-1)}(x'), \ldots, \pi_{k_f/m_f}(x^{(f)}))}{\partial(u_1, u_2, \ldots, u_{2f})} du_1 \, du_2 \ldots du_{2f} \\
&= \frac{1}{2^f} \sum \pm \{u_{\nu_1}, u_{\nu_2}\}\{u_{\nu_3}, u_{\nu_4}\} \ldots \{u_{\nu_{2f-1}}, u_{\nu_{2f}}\} \, du_1 \, du_2 \ldots du_{2f}.
\end{aligned}$$

The summation is extended over all the permutations $(\nu_1, \nu_2, \ldots, \nu_{2f})$ with the sign positive for even permutations and negative for odd permutations. Using (2.22) one obtains for the transformed integrals

$$\bar{\mathscr{T}}_f = \mathscr{T}/\tilde{c}^f \qquad (1 \leq f \leq rsN). \tag{2.33}$$

Consequently, all the generalised Poincaré integrals of higher order, which can be formed for each fixed value of N, remain invariant in the canonical transformations up to the multiplier $1/\tilde{c}^f$.

2.7. *Generalised Liouville theorem*

As a consequence of the invariance of the last integral (2.31), the corresponding Liouville theorem follows directly in the limit $N \to \infty$. Namely, for $f = rsN$ and $c = 1$ one has

$$\bar{\mathscr{T}}_{rsN} = \mathscr{T}_{rsN} \qquad (c = 1)$$

and this integral for $N \to \infty$ represents, according to (2.32), the volume of the considered phase space

$$\overline{\Delta\Gamma} = \Delta\Gamma, \tag{2.34}$$

which may be interpreted, in view of (2.15), in the following manner. Consider all the representative points in phase space occupying the volume $\Delta\Gamma_1 = \Delta\Gamma$ at the moment t_1; at some later moment t_2 they will occupy another part of the phase space of the same volume $\Delta\Gamma_2 = \overline{\Delta\Gamma}$, i.e.

$$\Delta\Gamma_2 = \Delta\Gamma_1. \tag{2.35}$$

Hence, the density of the representative points along their trajectories in the phase space remains constant with time, which can be expressed as

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = [\rho, H] + \frac{\partial\rho}{\partial t} = 0. \tag{2.36}$$

This is the corresponding generalised Liouville theorem, which is also seen to remain valid for classical fields in the considered case, and on this basis the usual statistical physics could be extended to such continuous systems.

3. Covariant formulation

3.1. *de Donder's equations in Weyl formalism*

Let us consider a physical field defined by r field functions $\psi^k = \psi^k(x^\alpha)$ and let this field be described by a certain Lagrangian, which depends also on derivatives up to order s:

$$\mathscr{L} = \mathscr{L}(\psi^k;\ \psi^k_{,\alpha_1}; \ldots;\ \psi^k_{,\alpha_1\ldots\alpha_s};\ x^\alpha), \tag{3.1}$$

where $\psi^k_{,\alpha_1} = \partial\psi^k/\partial x^{\alpha_1}$ and so on. The corresponding Lagrangian equations, equivalent to the Hamiltonian principle, have the form

$$\frac{\partial\mathscr{L}}{\partial\psi^k} - \frac{\mathrm{d}}{\mathrm{d}x^{\alpha_1}}\frac{\partial\mathscr{L}}{\partial\psi^k_{,\alpha_1}} + \frac{\mathrm{d}^2}{\mathrm{d}x^{\alpha_1}\,\mathrm{d}x^{\alpha_2}}\frac{\partial\mathscr{L}}{\partial\psi^k_{,\alpha_1\alpha_2}} - \ldots + (-1)^s\frac{\mathrm{d}^s}{\mathrm{d}x^{\alpha_1}\,\mathrm{d}x^{\alpha_2}\ldots\mathrm{d}x^{\alpha_s}}\frac{\partial\mathscr{L}}{\partial\psi^k_{,\alpha_1\ldots\alpha_s}} = 0. \tag{3.2}$$

For this case de Donder (1935) has introduced generalised momenta in his invariant theory of the calculus of variations, namely

$$\pi_k^{\alpha_1 \ldots \alpha_m} = \frac{\delta W}{\delta \psi^k_{,\alpha_1 \ldots \alpha_m}}, \qquad W \equiv \int \mathscr{L}\, \mathrm{d}^4 x \tag{3.3}$$

with four generalised momenta $\pi_k^{\alpha_1 \ldots \alpha_m}$ corresponding to each function $\psi^k_{,\alpha_1 \ldots \alpha_{m-1}}$, or explicitly

$$\pi_k^{\alpha_1 \ldots \alpha_m} = \sum_{j=0}^{s-m} (-1)^j \frac{\mathrm{d}^j}{\mathrm{d}x^{\alpha'_1}\, \mathrm{d}x^{\alpha'_2} \ldots \mathrm{d}x^{\alpha'_j}} \frac{\partial \mathscr{L}}{\partial \psi^k_{,\alpha_1 \ldots \alpha_m \alpha'_1 \ldots \alpha'_j}}. \tag{3.4}$$

On this basis he obtained the equations of the extremals in the canonical form

$$\frac{\mathrm{d}}{\mathrm{d}x^{\alpha_m}} \pi_k^{\alpha_1 \ldots \alpha_m} = -\frac{\partial \mathscr{H}}{\partial \psi^k_{,\alpha_1 \ldots \alpha_{m-1}}}, \qquad \frac{\mathrm{d}}{\mathrm{d}x^{\alpha_m}} \psi^k_{,\alpha_1 \ldots \alpha_{m-1}} = \frac{\partial \mathscr{H}}{\partial \pi_k^{\alpha_1 \ldots \alpha_m}},$$
$$(k = 1, 2, \ldots, r;\ m = 1, 2, \ldots, s) \tag{3.5}$$

where the Hamiltonian is

$$\mathscr{H}(\psi^k_{,\alpha_1 \ldots \alpha_{m-1}};\ \pi_k^{\alpha_1 \ldots \alpha_m};\ x^\alpha) = \pi_k^{\alpha_1 \ldots \alpha_m} \psi^k_{,\alpha_1 \ldots \alpha_m} - \mathscr{L}. \tag{3.6}$$

These equations are the corresponding generalised Hamiltonian equations and can be interpreted as the covariant canonical equations in Weyl formalism. They represent a system of $5r(4^s - 1)/3$ first-order differential equations with as many unknown functions

$$\psi^k,\ \psi^k_{,\alpha_1}, \ldots, \psi^k_{\alpha_1 \ldots \alpha_{m-1}}; \qquad \pi_k^{\alpha_1}, \pi_k^{\alpha_1 \alpha_2}, \ldots, \pi_k^{\alpha_1 \ldots \alpha_m},$$

which in this case play the role of the canonical variables.

3.2. *Total variation of the action*

For this considered case Borneas (1969) has studied the general variation of the action

$$\delta W = \int \delta \mathscr{L}\, \mathrm{d}^4 x + \int \mathscr{L} \delta(\mathrm{d}^4 x), \tag{3.7}$$

assuming that not only the functions $\psi^k_{,\alpha_1 \ldots \alpha_m}$ but also the independent variables x^α and the domain of integration vary. By decomposing the first variation into parts which arise from a change in the form of functions and from a change of independent variables, he found

$$\delta W = \int \frac{\delta W}{\delta \psi^k} \delta \psi^k\, \mathrm{d}^4 x + \int \frac{\mathrm{d}}{\mathrm{d}x^{\alpha_m}} (\pi_k^{\alpha_1 \ldots \alpha_m} \delta \psi^k_{,\alpha_1 \ldots \alpha_{m-1}} - T_\beta^{\alpha_m} \delta x^\beta)\, \mathrm{d}^4 x, \tag{3.8}$$

where

$$T_\beta^{\alpha_m} = \pi_k^{\alpha_1 \ldots \alpha_m} \psi^k_{,\alpha_1 \ldots \alpha_{m-1} \beta} - \delta_\beta^{\alpha_m} \mathscr{L}. \tag{3.9}$$

This set $T_\beta^{\alpha_m}$ represents the generalisation of the energy–momentum tensor in the field theory.

If the Lagrangian equations (3.2) are satisfied, the first term in (3.8) vanishes and the second term may be transformed into the surface integral with the aid of Gauss' theorem. In this way, choosing the variations on the time-like part of the surface σ to be zero, one obtains

$$\delta W = \left| \int (\pi_k^{\alpha_1 \ldots \alpha_m} \delta \psi^k_{,\alpha_1 \ldots \alpha_{m-1}} - T_\beta^{\alpha_m} \delta x^\beta) \, \mathrm{d}\sigma_{\alpha_m} \right|_{\sigma_1}^{\sigma_2}, \tag{3.10}$$

where σ_1 and σ_2 are the space-like parts of the surface σ.

This is the generalised 'boundary formula' in the Weyl formalism and is equivalent to the validity of the corresponding Lagrangian or Hamiltonian equations. Indeed, if these equations are satisfied, the relation (3.9) is valid, and vice versa.

3.3. *Formulation of the canonical transformations*

Let us consider the transformations of the canonical variables on one of two space-like surfaces σ_i $(i = 1, 2)$ in the form of the functionals

$$\begin{aligned} \bar{\psi}^k_{,\alpha_1 \ldots \alpha_{m-1}} &= F^k_{\alpha_1 \ldots \alpha_{m-1}}[\psi^l_{,\beta_1 \ldots \beta_{m-1}};\ \pi_l^{\beta_1 \ldots \beta_m};\ x^\beta] \\ \bar{\pi}_k^{\alpha_1 \ldots \alpha_m} &= G_k^{\alpha_1 \ldots \alpha_m}[\psi^l_{,\beta_1 \ldots \beta_{m-1}};\ \pi_l^{\beta_1 \ldots \beta_m};\ x^\beta]. \end{aligned} \tag{3.11}$$

If these transformations leave the form of the generalised Hamiltonian equations (3.5) invariant, we will call them canonical transformations.

The generalised Hamiltonian equations are equivalent to the boundary formula (3.10), which should also be valid for the new canonical variables, and so the necessary and sufficient condition for the transformation (3.11) to be canonical is

$$\begin{aligned} &\int (\pi_k^{\alpha_1 \ldots \alpha_m} \delta \psi^k_{,\alpha_1 \ldots \alpha_{m-1}} - T_\beta^{\alpha_m} \delta x^\beta) \, \mathrm{d}\sigma_{\alpha_m} \\ &\qquad = \int c(u^i)(\bar{\pi}_k^{\alpha_1 \ldots \alpha_m} \delta \bar{\psi}^k_{,\alpha_1 \ldots \alpha_{m-1}} - \bar{T}_\beta^{\alpha_m} \delta x^\beta) \, \mathrm{d}\sigma_{\alpha_m} + \delta G. \end{aligned} \tag{3.12}$$

Here $c(u^i)$ is any function of the parameters u^i, which define the position on the surface σ, and G is the corresponding generating functional. If we transform this condition and introduce the quantities

$$\begin{aligned} G_1[\psi^k_{,\alpha_1 \ldots \alpha_{m-1}};\ \bar{\pi}_k^{\alpha_1 \ldots \alpha_m};\ x^\alpha] &= G + \int c \bar{\pi}_k^{\alpha_1 \ldots \alpha_m} \bar{\psi}^k_{,\alpha_1 \ldots \alpha_{m-1}} \, \mathrm{d}\sigma_{\alpha_m} \\ K_\beta &= \int T_\beta^{\alpha_m} \, \mathrm{d}\sigma_{\alpha_m}, \end{aligned} \tag{3.13}$$

the comparison of the corresponding coefficients yields

$$\begin{aligned} &\pi_k^{\alpha_1 \ldots \alpha_m} = n^{\alpha_m} \frac{\delta G_1}{\delta \psi^k_{,\alpha_1 \ldots \alpha_{m-1}}}, \qquad c \bar{\psi}^k_{,\alpha_1 \ldots \alpha_{m-1}} = n^{\alpha_m} \frac{\delta G_1}{\delta \bar{\pi}_k^{\alpha_1 \ldots \alpha_m}} \\ &\tilde{c} \bar{K}_\beta = K_\beta + \frac{\partial G_1}{\partial x^\beta}, \end{aligned} \tag{3.14}$$

where n^{α_m} are the contravariant components of the unit vector of the normal and $\tilde{c}$ is the mean value of the function $c(u^i)$ on σ_i. Similarly,

$$\begin{aligned} &\psi^k_{,\alpha_1 \ldots \alpha_{m-1}} = -n^{\alpha_m} \frac{\delta G_2}{\delta \pi_k^{\alpha_1 \ldots \alpha_m}}, \qquad c \bar{\pi}_k^{\alpha_1 \ldots \alpha_m} = -n^{\alpha_m} \frac{\delta G_2}{\delta \bar{\psi}^k_{,\alpha_1 \ldots \alpha_{m-1}}} \\ &\tilde{c} \bar{K}_\beta = K_\beta + \frac{\partial G_2}{\partial x^\beta}. \end{aligned} \tag{3.15}$$

The discussion of this system of equations is similar to that in classical field theory. Namely, this is a system of differential or functional differential equations, depending on whether or not the generating functional has the form of an integral, and by solving this system of equations with respect to $\bar{\pi}_k^{\alpha_1\ldots\alpha_m}$, or $\bar{\psi}_{,\alpha_1\ldots\alpha_{m-1}}$, we obtain new canonical variables as functionals of the old ones.

For the canonical transformation of the form

$$\bar{\psi}^k_{,\alpha_1\ldots\alpha_{m-1}} = \psi^k_{,\alpha_1\ldots\alpha_{m-1}} + \delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}, \qquad \bar{\pi}_k^{\alpha_1\ldots\alpha_m} = \pi_k^{\alpha_1\ldots\alpha_m} + \delta\pi_k^{\alpha_1\ldots\alpha_m} \tag{3.16}$$

one may take as the generating functional

$$G_1 = \int \psi^k_{,\alpha_1\ldots\alpha_{m-1}} \bar{\pi}_k^{\alpha_1\ldots\alpha_m}\, \mathrm{d}\sigma_{\alpha_m} + \epsilon G'[\psi^k_{,\alpha_1\ldots\alpha_{m-1}};\ \bar{\pi}_k^{\alpha_1\ldots\alpha_m};\ x^\alpha], \tag{3.17}$$

where ϵ is a small parameter. On the basis of (3.14) one then obtains approximately

$$\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}} = \epsilon n^{\alpha_m} \frac{\delta G'}{\delta\pi_k^{\alpha_1\ldots\alpha_m}}, \qquad \delta\pi_k^{\alpha_1\ldots\alpha_m} = -\epsilon n^{\alpha_m} \frac{\delta G'}{\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}}. \tag{3.18}$$

With the aid of these relations, the variation of any functional of the canonical variables may be found:

$$\delta F = F[\bar{\psi}^k_{,\alpha_1\ldots\alpha_{m-1}};\ \bar{\pi}_k^{\alpha_1\ldots\alpha_m};\ x^\alpha] - F[\psi^k_{,\alpha_1\ldots\alpha_{m-1}};\ \pi_k^{\alpha_1\ldots\alpha_m};\ x^\alpha].$$

Developing the first term into a Taylor series, generalised for the functionals, we find

$$\delta F = \epsilon [F, G'], \tag{3.19}$$

where the symbol [ ] denotes the generalised Poisson bracket

$$[F, G] = \int \left(\frac{\delta F}{\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}} \frac{\delta G}{\delta\pi_k^{\alpha_1\ldots\alpha_m}} - \frac{\delta F}{\delta\pi_k^{\alpha_1\ldots\alpha_m}} \frac{\delta G}{\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}} \right) \mathrm{d}\sigma^{\alpha_m}. \tag{3.20}$$

That is the generalisation of Freistadt's (1959) bracket, extended to this case.

3.4. *Hamilton–Jacobi method*

Let us consider now a canonical transformation for which the new Hamiltonian vanishes. In this case the new canonical variables are constants of motion in the sense

$$\partial\bar{\pi}_k^{\alpha_1\ldots\alpha_m}/\partial x^{\alpha_m} = 0, \qquad \partial\bar{\psi}^k_{,\alpha_1\ldots\alpha_{m-1}}/\partial x^{\alpha_m} = 0. \tag{3.21}$$

To obtain an equation determining the generating functional of this canonical transformation, we must express $T_\beta^{\alpha_m}$ as a function of $\mathscr{H}$ and find K_β according to (3.13):

$$K_\beta = \int \pi_k^{\alpha_1\ldots\alpha_m}(n_{\alpha_m}\psi^k_{,\alpha_1\ldots\alpha_{m-1}\beta} - n_\beta\psi^k_{,\alpha_1\ldots\alpha_m})\, \mathrm{d}\sigma + \int \mathscr{H}\, \mathrm{d}\sigma_\beta. \tag{3.22}$$

Since $\bar{\mathscr{H}} = 0$ in the new variables, and from (3.21) $\bar{\psi}^k_{,\alpha_1\ldots\alpha_{m-1}} = 0$, it follows that

$$\bar{K}_\beta = \int \bar{T}_\beta^{\alpha_m}\, \mathrm{d}\sigma_{\alpha_m} = 0.$$

If all the generalised momenta $\pi_k^{\alpha_1\ldots\alpha_m}$ in the expression for the functional K_β are substituted according to the first relation of (3.14) by $n_{\alpha_m}\delta S/\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}$, the last relation of (3.14) can be written in the form

$$n_{\alpha_m}\frac{\partial S^{\alpha_m}}{\partial x^\beta}+K_\beta\left[\psi^k_{,\alpha_1\ldots\alpha_{m-1}};\ \frac{\delta S^{\alpha_m}}{\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}};\ x^\alpha\right]=0, \tag{3.23}$$

where

$$S^{\alpha_m}=n^{\alpha_m}S. \tag{3.24}$$

These four equations are the corresponding Hamilton–Jacobi equations in this covariant formalism. To each coordinate x^β there corresponds a functional differential equation of this type, where one of the functionals K_β appears instead of the Hamiltonian, which constitutes a characteristic difference compared with classical field theory. For the space-like surface $t=\text{constant}$, $K_4=H$, so that the fourth equation is reduced to the Hamilton–Jacobi equation of classical field theory.

Let us assume that we have found a complete integral of this equation of the form

$$S^{\alpha_m}=S^{\alpha_m}[\psi^k_{,\alpha_1\ldots\alpha_{m-1}};\ a_k^{\alpha_1\ldots\alpha_m};\ x^\alpha],$$

where $a_k^{\alpha_1\ldots\alpha_m}$ are arbitrary functions of x^α with the divergence being equal to zero. Taking $a_k^{\alpha_1\ldots\alpha_m}=\bar{\pi}_k^{\alpha_1\ldots\alpha_m}$, one obtains all the canonical variables by solving the corresponding system of equations, differential or functional

$$\begin{aligned}\frac{\delta S^{\alpha_m}[\psi^k_{,\alpha_1\ldots\alpha_{m-1}};\ a_k^{\alpha_1\ldots\alpha_m};\ x^\alpha]}{\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}}&=\pi_k^{\alpha_1\ldots\alpha_m}\\ \frac{\delta S^{\alpha_m}[\psi^k_{,\alpha_1\ldots\alpha_{m-1}};\ a_k^{\alpha_1\ldots\alpha_m};\ x^\alpha]}{\delta a_k^{\alpha_1\ldots\alpha_m}}&=cb^k_{\alpha_1\ldots\alpha_{m-1}}\end{aligned} \tag{3.25}$$

with respect to $\psi^k_{,\alpha_1\ldots\alpha_{m-1}}$ and $\pi_k^{\alpha_1\ldots\alpha_m}$.

3.5. *Lagrange and Poisson brackets*

Let us express the condition (3.12) as a function of primary variables only, substituting $\delta\bar{\psi}^k_{,\alpha_1\ldots\alpha_{m-1}}$ by the corresponding functional differential. Using the conditions that this expression be a total differential, in a manner similar to that of classical field theory, one obtains

$$\begin{aligned}&\{\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x),\ \psi^l_{,\beta_1\ldots\beta_{m-1}}(x')\}_{\bar\psi,\bar\pi}=0,\qquad \{\pi_k^{\alpha_1\ldots\alpha_m}(x),\ \pi_l^{\beta_1\ldots\beta_m}(x')\}_{\bar\psi,\bar\pi}=0\\ &\{\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x),\ \pi_l^{\beta_1\ldots\beta_m}(x')\}_{\bar\psi,\bar\pi}=\delta^\alpha_\beta\delta^l_k\delta_{\beta_m}(x-x')/\tilde{c},\end{aligned} \tag{3.26}$$

where the generalised Lagrangian bracket is introduced as:

$$\{u(x),v(x')\}=\int\left(\frac{\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x'')}{\delta u(x)}\frac{\delta\pi_k^{\alpha_1\ldots\alpha_m}(x'')}{\delta v(x')}-\frac{\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x'')}{\delta v(x')}\frac{\delta\pi_k^{\alpha_1\ldots\alpha_m}(x'')}{\delta u(x)}\right)\mathrm{d}\sigma''_{\alpha_m} \tag{3.27}$$

and where $\delta_{\beta_m}(x-x')=n_{\beta_m}\delta(x-x')$. These conditions may be formulated also using generalized Poisson brackets, in the form:

$$\begin{aligned}&[\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x),\ \psi^l_{,\beta_1\ldots\beta_{m-1}}(x')]_{\bar\psi,\bar\pi}=0,\qquad [\pi_k^{\alpha_1\ldots\alpha_m}(x),\ \pi_l^{\beta_1\ldots\beta_m}(x')]_{\bar\psi,\bar\pi}=0\\ &[\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x),\ \pi_l^{\beta_1\ldots\beta_m}(x')]_{\bar\psi,\bar\pi}=\tilde{c}\delta^\beta_\alpha\delta^k_l\delta^{\beta_m}(x-x').\end{aligned} \tag{3.28}$$

If the generalised Lagrange bracket is formed in new canonical variables, and then transformed back to the original variables, the result can be expressed by the fundamental brackets. In this way, on the basis of (3.26), one obtains

$$\{u(x), v(x')\}_{\bar{\psi},\bar{\pi}} = \{u(x), v(x')\}_{\psi,\pi}/\tilde{c} \tag{3.29}$$

and by a similar procedure for the generalised Poisson bracket:

$$[F, G]_{\bar{\psi},\bar{\pi}} = \tilde{c}[F, G]_{\psi,\pi}. \tag{3.30}$$

Then the generalised Lagrange and Poisson brackets remain invariant under canonical transformations up to the multiplier $1/\tilde{c}$, or $\tilde{c}$.

3.6. Canonical equations in the form of Poisson brackets

Let us form the generalised Poisson brackets with canonical variables for any functional of the form $F[\psi^k_{,\alpha_1\ldots\alpha_{m-1}};\ \pi_k^{\alpha_1\cdots\alpha_m};\ x^\alpha]$. Thus, bearing in mind the properties of Dirac's delta function, one obtains

$$[\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x), F] = n^{\alpha_m}(x)\,\delta F/\delta\pi_k^{\alpha_1\cdots\alpha_m}(x) \tag{3.31}$$

and similarly, multiplying by $n_{\alpha_m(x)}$

$$n_{\alpha_m(x)}[\pi_k^{\alpha_1\cdots\alpha_m}(x), F] = -\delta F/\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x). \tag{3.32}$$

If one takes $F = H$ and the expressions on the right-hand side are substituted according to the generalised Hamiltonian equations, the above relations assume the form

$$\begin{aligned} n^{\alpha_m}(x)\frac{\mathrm{d}\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x)}{\mathrm{d}x^{\alpha_m}} &= [\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x), H], \\ \frac{\mathrm{d}\pi_k^{\alpha_1\cdots\alpha_m}(x)}{\mathrm{d}x^{\alpha_m}} &= n_{\alpha_m}(x)[\pi_k^{\alpha_1\cdots\alpha_m}(x), H]. \end{aligned} \tag{3.33}$$

These are the generalised Hamiltonian equations, i.e. de Donder's equations in the form of Poisson brackets. They might be of special interest in quantum mechanics for the quantisation of physical fields. If the analogy between classical and quantum mechanics remains valid for this generalised case, then the transition to quantum equations may be made effectively by substituting the corresponding commutators for these generalised Poisson brackets.

3.7. Integral invariants of the first order

In order to represent geometrically the states of a physical field in a certain point x^β of Minkowski space, let us introduce the phase space as a Euclidean space with elements $\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x^\beta)$ and $\pi_k^{\alpha_1\cdots\alpha_m}(x^\beta)$. Now let us consider, in the enlarged phase space, any closed curve L_0, defined by

$$\psi^k_{,\alpha_1\ldots\alpha_{m-1}} = \psi^k_{,\alpha_1\ldots\alpha_{m-1}}(w, \lambda), \qquad \pi_k^{\alpha_1\ldots\alpha_m} = \pi_k^{\alpha_1\ldots\alpha_m}(w, \lambda), \tag{3.34}$$

where w is the time-like parameter defining the surface σ, and through every point of that curve for constant w, we have the corresponding trajectory of the state. Then the corresponding integral invariants may be obtained by the following procedure.

If we begin with the boundary formula (3.10) and assume that these variations arise from the change of the parameter λ, then integrating this relation along the contour L_0 gives

$$\begin{aligned}\mathcal{T} &\equiv \oint_{L_0}\int (\pi_k^{\alpha_1\ldots\alpha_m}\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}} - T_\beta^{\alpha_m}\delta x^\beta)\,\mathrm{d}\sigma_{\alpha_m} \\ &= \oint_{L_1}\int (\pi_k^{\alpha_1\ldots\alpha_m}\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}} - T_\beta^{\alpha_m}\delta x^\beta)\,\mathrm{d}\sigma_{\alpha_m}.\end{aligned} \tag{3.35}$$

This integral represents the generalisation of the Poincaré–Cartan integral invariant to the covariant field theory, and the above relation expresses its invariance along any contour on the surface formed by the state trajectories.

Let us start now from the necessary and sufficient condition (3.12) and integrate this relation along any contour. Since $\oint \delta G = 0$, we find

$$\bar{\mathcal{T}} = \mathcal{T}/\tilde{c}, \tag{3.36}$$

i.e. the generalised Poincaré–Cartan integral is invariant with respect to the canonical transformations up to the multiplier $1/\tilde{c}$.

These integral invariants differ from those of classical field theory in that the tensor $T_\beta^{\alpha_m}$ appears instead of the Hamiltonian, and the integration is carried out over the space-like surface σ_i instead of the volume. This is at the same time the essential difference from the integral invariants found by de Donder (1935, p 106). Taking t to be constant for the space-like surface, the generalised Poincaré–Cartan integral invariant is reduced to that of classical field theory.

3.8. *Absolute integral invariants of higher order*

The integral (3.35) for $\delta x^\beta = 0$ may be transformed into a surface integral with the aid of Stokes' theorem for the functionals

$$\mathcal{T}_1 \equiv \oint_L \int \pi_k^{\alpha_1\ldots\alpha_m}\delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}\,\mathrm{d}\sigma_{\alpha_m} = \int_S\int \delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}\delta\pi_k^{\alpha_1\ldots\alpha_m}\,\mathrm{d}\sigma_{\alpha_m}. \tag{3.37}$$

In order to obtain other higher-order integral invariants, let us divide the domain σ_i into a very large number N of cells, as in classical field theory, taking the values of the functions $\psi^k(x^\alpha)$ to be equal at every point of the same cell. By this procedure the considered physical field is substituted by the equivalent system of Nr degrees of freedom.

For a fixed value of N let us form the following set of integrals:

$$\begin{aligned}\mathcal{T}_1 &= \int_S \sum_x \delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x)\delta\pi_k^{\alpha_1\ldots\alpha_m}(x) \\ \mathcal{T}_2 &= \int_S\int_S \sum_{x,x'} \delta\psi^k_{,\alpha_1\ldots\alpha_{m-1}}(x)\delta\psi^{k'}_{,\alpha'_1\ldots\alpha'_{m-1}}(x')\delta\pi_k^{\alpha_1\ldots\alpha_m}(x)\delta\pi_{k'}^{\alpha'_1\ldots\alpha'_m}(x') \\ &\vdots \\ \mathcal{T}_{rsN} &= \int_S\int_S\cdots\int_S \delta\psi^1(x_1)\delta\psi^1(x_2)\ldots\delta\psi^r_{,\alpha_1\ldots\alpha_{s-1}}(x_N)\delta\pi_1^1(x_1)\delta\pi_1^1(x_2)\ldots\delta\pi_r^{\alpha_1\ldots\alpha_s}(x_N),\end{aligned} \tag{3.38}$$

where Σ_x denotes the summation over the quoted cells and where the summation has been carried out only with respect to terms with different indices. The invariance of these integrals may be proved by introducing Gaussian parameters as integration variables and by generalising the corresponding relation from analytical mechanics to this case. In this way the integrand of each of these integrals may be expressed by the generalised Lagrange brackets, and thus on the basis of (3.29) one obtains

$$\bar{\mathcal{T}}_f = \mathcal{T}_f/\tilde{c}^f \qquad (1 \leqslant f \leqslant rsN). \tag{3.39}$$

Hence, all the generalised Poincaré integrals of higher order for any fixed value of N remain invariant up to the multiplier $1/\tilde{c}^f$. Nevertheless, this result does not enable us to express, within this covariant formalism, the corresponding Liouville theorem by use of only space–time coordinates x^α, while this is possible in the parametric formalism.

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